

INTRODUCTION: Special Focus Issue

Developing Algebraic Thinking in the Earlier Grades from an International Perspective¹

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Algebra readiness has been characterized as the most important "gatekeeper" in mathematics. It is widely accepted that to achieve the goal of "algebra for all," students in elementary school should have experiences that prepare them for more formal study of algebra in the later grades (e.g., NCTM, 2000). However, curriculum developers, educational researchers, and policy makers are just beginning to think about and explore the kinds of mathematical experiences elementary students need to prepare them for the formal study of algebra in the later grades (e.g., Carpenter, Franke, & Levi, 2003; Kaput, 1999; Kieran, 1996; Mathematical Sciences Education Board, 1998; NCTM, 2000; Schifter, 1999). This Special Issue of the journal provides an international perspective of developing students' informal and formal algebraic thinking in elementary school. Case studies of the intended elementary mathematics curricula in China, Russia, South Korea, Singapore, and the United States are presented to show the ways different curricula provide students with informal and formal algebraic experiences. The findings from these case studies should help educators and curriculum developers establish a mechanism that provides students experiences with both algebraic ideas and thinking in the earlier grades. While any curriculum has a complex relationship with what actually occurs in classrooms, research has shown that curriculum substantially determines the course of instruction and learning in school (Schmidt, et al., 1996). Therefore, we believe that these case studies may also contribute to our understanding of the impact of curriculum and instruction on students' learning and understanding.

Each case study focuses on how the elementary curriculum in a particular country has been designed to develop students' algebraic thinking: Specifically, each case addresses what algebraic concepts are included in the curriculum, and how these algebraic concepts and representations are introduced. In particular, in each case

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study, a curriculum is analyzed in three dimensions: (1) goal specification, (2) content coverage, and (3) process coverage.

Goal Specification. In this dimension of the analysis, algebra-related goals in each curriculum are identified, as are a set of specific mathematical problems in each curriculum. Solving these problems is considered an indication of reaching the goals. The identified algebra goals in each curriculum are compared with the four algebra goals specified in the National Council of Teachers of Mathematics' (NCTM) Principles and Standards for School Mathematics (NCTM, 2000). These four goals are to (1) understand patterns, relations, and functions; (2) represent and analyze mathematical situations and structures using algebraic symbols; (3) use mathematical models to represent and understand quantitative relationships; and (4) analyze change in various contexts.

Content Coverage. For the second dimension, BIG IDEAS of algebraic thinking in each of the mathematics curricula are identified. A big idea of algebraic thinking is an essential concept or technique for reasoning about quantitative conditions and relationships. In these case studies, we have focused on the following commonly identified algebraic ideas: variables, proportional reasoning, patterns and relationships, equivalence of expressions, equation and equation solving, change, and representation and modeling, which are widely accepted as important in algebra. In each case study, the ways some of the big ideas develop throughout a curriculum are also examined.

Process Coverage. Algebra is much more than just solving for x and y ; instead, algebra is a way of thinking. Success in algebra depends on the ability to think in a variety of powerful ways that foster productive algebraic performance. When people think algebraically to solve problems, various habits of thinking come into play, such as Doing-Undoing, Building Rules to Represent Functions, and Abstracting from Computation (Driscoll, 1999). Curricula can serve to demystify algebra by providing activities that foster these sorts of thinking in students. In this third dimension of analysis, we examine how a curriculum is designed to develop algebraic thinking habits. In this dimension, a case study also includes the examination of how each of the curricula helps students make the connection between the way these habits of thinking are employed in their pre-algebra experience, and the way they are employed when doing formal algebra.

It should be indicated that due to the variations of the curricula analyzed, the case studies do not follow the three dimensions in exactly the same way. Thus, the results from these case studies are not presented in identical formats. It should also be noted that the case studies in this Special Issue are not intended to evaluate the

curricula in these countries. Instead, our focus is on studying and understanding how curricula in these countries are designed to develop students' algebraic thinking. These case studies provide an international perspective of the kinds of algebraic experience elementary school students should have. In addition, to support students' development of algebraic thinking, we need to help them make a smooth transition between arithmetic and algebraic thinking and to appreciate the usefulness of algebraic approaches in solving various problems. These case studies provide in-depth information about the development of algebraic thinking in five countries, with special attention to the transition from informal to formal algebraic thinking.

From these case studies, readers will find that all of the curricula explicitly or implicitly indicate that the main goal for learning algebraic concepts is to deepen students' understanding of quantitative relationships. However, the emphases and approaches to help students deepen their understanding of quantitative relationships are very different across the five curricula. Let us note briefly some features of these five curricula.

For the U.S. curriculum, the *Investigations* in Number, Data, and Space curriculum has been chosen for the analysis. It is one of the three elementary mathematics reform curricula and it is a widely used series in the United States. "Analyze change in its various contexts" is the central goal of the *Investigations'* algebra strand. A strength of the *Investigations* curriculum is its approach to achieving one of its overarching goals "to emphasize depth in mathematical thinking rather than superficial exposure to a series of fragmented topics." This is seen in the algebra strand through the use of the concept of change to unify students' understanding of different algebraic ideas, concepts, representations, and models. This curriculum accomplishes the teaching of "change" from an informal perspective. The curriculum builds on basic intuitions that children express from a very early age, enriching and refining them. By design, therefore, it does not progress to a symbolic or formal work.

In Singapore, the elementary school curriculum provides varied resources for developing students' algebraic thinking. Students have ample opportunities to make generalizations through number pattern activities. Equations are not introduced symbolically; instead, they introduced through pictures. Such "pictorial equations" are used extensively to represent quantitative relationships. The pictorial equations not only provide a tool for students to solve mathematical problems, but also provide a means to develop students' algebraic ideas.

The Russian curriculum of Davydov engages children in a focused analysis and description of the quantitative world as the starting point for developing algebraic

thought. The curriculum emphasizes the development of algebraic understandings through direct work with quantities, and the representational modeling of mathematical actions and relations. This manipulation of quantities (e.g., constructing and measuring quantities) lays the foundation for ideas about mathematical relationships and generalizations. The analysis of relationships between quantities that the students do with schematics and equations not only engenders a meaning for equations, but also enables students to reason algebraically in a way that a focus on only the numerical values of the sets does not. The inductive discovery aspect of the Algebra Standard is absent in the Russian curriculum, and there is no work with patterns.

In South Korea, students begin the formal algebra course in the 7th grade. In elementary school, many concrete operational activities are used to reduce the cognitive gap between algebra and arithmetic. For example, the symbol “?” to represent an unknown value is introduced at the first grade level, and an intuitive process to solve for the unknown value “?” is also introduced. In grades 3 and 4, the process of solving equations more formally by using inverse operations is introduced.

In the Chinese curriculum, the main focus is on equation and equation solving. Variables, equations, equation solving, and function sense permeate the arithmetic analysis of quantitative relationships in the curriculum for grades 1 to 4. Equations and equation solving are formally introduced in the first half of grade 5. The Chinese elementary school curriculum emphasizes the examination of quantitative relationships from different perspectives. Students are consistently encouraged and provided with opportunities to represent a quantitative relationship both arithmetically and algebraically. Furthermore, students are asked to make comparisons between arithmetical and algebraic ways of representing a quantitative relationship.

The presentation of the five case studies is followed by two commentaries. These two commentaries discuss the five case studies from different perspectives. In the first commentary, Beverly Ferrucci uses the National Assessment of Educational Progress algebra framework to summarize the similarities and differences of the five curricula described in this Special Issue (National Assessment Governing Board, 2001). In the second commentary, Carolyn Kieran situates her discussion of algebraic thinking in the earlier grades, in general, and this set of five case studies, in specific, in an existing model of algebraic thinking she proposes for later grades (Kieran, 1996).

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